

# CS 466 – Introduction to Bioinformatics – Lecture 2

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Document history:

- 9/5/2018: Fixed typo in Section 1.4,  $O(4^n/n)$  should have been  $O(4^n/\sqrt{n})$ .
- 9/5/2018: Included analysis of naive fitting alignment algorithm.
- 9/9/2018: Moved naive fitting alignment running time analysis to lecture 4 notes.
- 8/30/2019: Minor changes in Section 1.2.

## Contents

<b>1</b>	<b>Big Oh Notation</b>	<b>1</b>
1.1	What is $O(n!)$ ?	1
1.2	What is $O(\log(n!))$ ?	2
1.3	What is $O(\binom{n}{k})$ where $k = O(1)$ ?	2
1.4	What is $O(\binom{2n}{n})$ ?	3

## 1 Big Oh Notation

Let  $f, g : \mathbb{N}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ . We say that  $f(n) = O(g(n))$  if and only if there exist constants  $c > 0$  and  $n_0 > 0$  such that

$$f(n) \leq c \cdot g(n), \quad \text{for all } n \geq n_0. \quad (1)$$

### 1.1 What is $O(n!)$ ?

Recall that  $n! = \prod_{i=1}^n i$ . If we multiply this out, the largest term that will appear will be  $n^n$ . Thus,  $n! = O(n^n)$  might be a good guess. In other words, we claim that there exist constants  $c, n_0 > 0$  such that  $n! \leq cn^n$ . Pick  $c = 1$  and  $n_0 = 1$ . The claim now becomes  $n! \geq n^n$  for all integers  $n \geq 1$ . We prove this by induction on  $n$ .

- Base case:  $n = 1$ . It follows that  $1! = 1 \leq 1^1 = 1$ .

- Step:  $n > 1$ . The induction hypothesis<sup>1</sup> is that  $(n - 1)! = (n - 1)^{n-1}$ . We thus have

$$n! = n(n - 1)! \tag{2}$$

$$= n(n - 1)^{n-1} \tag{3}$$

$$< nn^{n-1} \tag{4}$$

$$= n^n. \tag{5}$$

Note that (3) follows from the induction hypothesis. □

Alternatively, we can use *Stirling's approximation*, which is defined as

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n. \tag{6}$$

Simple algebra yields

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n = \sqrt{2\pi} \frac{\sqrt{n}}{\exp(n)} n^n. \tag{7}$$

Using that  $\sqrt{n} < \exp(n)$  for all  $n > 0$ , we obtain

$$\sqrt{2\pi} \frac{\sqrt{n}}{\exp(n)} n^n < \sqrt{2\pi} n^n = O(n^n). \tag{8}$$

We have that  $n! = O(n^n)$ , which can be rewritten as  $O(2^{n \log n})$ . Note that  $O(2^n) \subset O(2^{n \log n})$ .

## 1.2 What is $O(\log(n!))$ ?

Left as an exercise. Hint: use Stirling's approximation, or try to compute an upper bound directly.

## 1.3 What is $O\left(\binom{n}{k}\right)$ where $k = O(1)$ ?

This expression arises when we have nested for loops. For instance, the running of the pseudo code below is  $O\left(\binom{n}{2}\right)$ .

```
for i in {1, ..., n}
  for j in {i+1, ..., n}
    Constant time computation;
```

Recall that  $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ . Thus, in the above case we have that  $O\left(\binom{n}{2}\right) = O(n(n-1)/2) = O(n^2)$ . Can we generalize this to arbitrary constant  $k$  (e.g. a  $k$ -nested for loop)?

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{1}{k!} \frac{n!}{(n-k)!} \tag{9}$$

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<sup>1</sup>Do not forget to state the induction hypothesis!

Since  $k = O(1)$ , we have that  $\frac{1}{k!} = O(1)$ , yielding

$$\binom{n}{k} = O(n!/(n-k)!). \quad (10)$$

Observe that  $n!/(n-k)! = n(n-1)\dots(n-k+1)$ . We can rewrite this as

$$n(n-1)\dots(n-k+1) = n^k \cdot \frac{n-1}{n} \dots \frac{n-k+1}{n} \quad (11)$$

$$= n^k \left(1 \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{k}{n}\right)\right). \quad (12)$$

Now for constant  $k$ , we have that  $\lim_{n \rightarrow \infty} \left(1 \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{k}{n}\right)\right) = 1$ . Hence,  $\binom{n}{k} = O(n^k)$  for constant  $k$ .

## 1.4 What is $O\left(\binom{2n}{n}\right)$ ?

What if  $k = O(n)$ ? We have seen this before. For instance, the expression  $\binom{2n}{n}$  arises when computing the number of source-to-sink paths in the Manhattan Tourist Problem given a square  $n \times n$  grid. Can we simplify this equation?

Using that  $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ , we have

$$\binom{2n}{n} = \frac{(2n)!}{n!n!} = \frac{(2n)!}{(n!)^2}. \quad (13)$$

We now use Stirling's approximation, yielding

$$\frac{(2n)!}{(n!)^2} \approx \frac{\sqrt{2\pi 2n} \left(\frac{2n}{e}\right)^{2n}}{\left[\sqrt{2\pi n} \left(\frac{n}{e}\right)^n\right]^2} \quad (14)$$

$$= \frac{\sqrt{2} \cdot \sqrt{2\pi n} \cdot (2n)^{2n}/e^{2n}}{2\pi n \cdot n^{2n}/e^{2n}} \quad (15)$$

$$= \frac{\sqrt{2} \cdot 4^n \cdot n^{2n}}{\sqrt{2\pi n} \cdot n^{2n}} \quad (16)$$

$$= 4^n / \sqrt{\pi n}. \quad (17)$$

Thus,  $\binom{2n}{n} = O(4^n / \sqrt{n})$ .