Course Announcements

Instructor:
• Mohammed El-Kebir (melkebir)
• Office hours: Wednesdays, 3:15-4:15pm

Zoom:
• https://illinois.zoom.us/j/93763998549

TA:
• Sarah Christensen (sac2), office hours: Mondays, 3-4pm via Zoom
• Wesley Wei Qian (weiqian3), office hours: Fridays, 9-10am via Zoom

Piazza: (please sign up)
• https://piazza.com/illinois/fall2020/cs466
Outline

1. Change problem
2. Review of running time analysis
3. Edit distance
4. Review elementary graph theory
5. Manhattan Tourist problem
6. Longest/shortest paths in DAGs

Reading:
• Jones and Pevzner. Chapters 2.7-2.9 and 6.1-6.4
• Lecture notes
The Change Problem

**Change Problem:** Given amount $M \in \mathbb{N} \setminus \{0\}$ and coins $c = (c_1, \ldots, c_n) \in \mathbb{N}^n$ s.t. $c_n = 1$ and $c_i \geq c_{i+1}$ for all $i \in [n - 1] = \{1, \ldots, n - 1\}$, find $d = (d_1, \ldots, d_n) \in \mathbb{N}^n$ s.t. (i) $M = \sum_{i=1}^{n} c_i d_i$ and (ii) $\sum_{i=1}^{n} d_i$ is minimum.

- Suppose we have $n = 3$ coins:

$$c = (\begin{array}{c} \text{7 cent} \\ \text{3 cent} \\ \text{1 cent} \end{array})$$

- What is the minimum number of coins needed to make change for $M = 9$ cents?

- Answer: $(d_1, \ldots, d_n) = (1, 0, 2)$ thus $1 + 0 + 2 = 3$ coins.
The Change Problem – Four Algorithms

GreedyChange\( (M, c_1, ..., c_n) \)

1. \( \text{for } i \leftarrow 1 \text{ to } n \)
2. \( d_i \leftarrow \lfloor M/c_i \rfloor \)
3. \( M \leftarrow M - d_i c_i \)

ExhaustiveChange\( (M, c_1, ..., c_n) \)

1. \( \text{for } (d_1, \ldots, d_n) \in \{0, \ldots, \lfloor M/c_1 \rfloor \} \times \ldots \times \{0, \ldots, \lfloor M/c_n \rfloor \} \)
2. \( \text{if } \sum_{i=1}^{n} c_i d_i = M \)
3. \( \text{return } (d_1, \ldots, d_n) \)

RecursiveChange\( (M, c_1, ..., c_n) \)

1. \( \text{if } M = 0 \)
2. \( \text{return } 0 \)
3. \( \text{bestNumCoins } \leftarrow \infty \)
4. \( \text{for } i \leftarrow 1 \text{ to } n \)
5. \( \text{if } M \geq c_i \)
6. \( \text{numCoins } \leftarrow \text{RecursiveChange}(M - c_i, c_1, ..., c_n) \)
7. \( \text{if } \text{numCoins} + 1 < \text{bestNumCoins} \)
8. \( \text{bestNumCoins } \leftarrow \text{numCoins} + 1 \)
9. \( \text{return } \text{bestNumCoins} \)

DPChange\( (M, c_1, ..., c_n) \)

1. \( \text{for } m \leftarrow 1 \text{ to } M \)
2. \( \text{minNumCoins}[m] \leftarrow \infty \)
3. \( \text{for } i \leftarrow 1 \text{ to } n \)
4. \( \text{minNumCoins}[c_i] \leftarrow 1 \)
5. \( \text{for } m \leftarrow 1 \text{ to } M \)
6. \( \text{for } i \leftarrow 1 \text{ to } n \)
7. \( \text{if } m > c_i \)
8. \( \text{minNumCoins}[m] \leftarrow \min(1 + \text{minNumCoins}[m - c_i], \text{minNumCoins}[m]) \)
9. \( \text{return } \text{minNumCoins}[M] \)
Four Different Algorithms

<table>
<thead>
<tr>
<th>Technique</th>
<th>Correct?</th>
<th>Efficient?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Greedy algorithm [GreedyChange]</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>Exhaustive enumeration [ExhaustiveChange]</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Recursive algorithm [RecursiveChange]</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>Dynamic programming [DPChange]</td>
<td>yes</td>
<td>yes</td>
</tr>
</tbody>
</table>

**Question:** How to assess efficiency?
Running Time Analysis

• The **running time** of an algorithm \( A \) for problem \( \Pi \) is the maximum number of steps that \( A \) will take on any instance of size \( n = |X| \)

• **Asymptotic running time** ignores constant factors using Big O notation

\[
f(n) \text{ is } O(g(n)) \text{ provided there exists } c > 0 \text{ and } n_0 \geq 0 \text{ such that } f(n) \leq c \cdot g(n) \text{ for all } n \geq n_0
\]
Running Time Analysis – Example

\[ f(n) = 10000 + 500n \]

\[ g(n) = \frac{n^4}{2} \]

\[ f(n) \text{ is } O(g(n)) \text{ provided there exists } c > 0 \text{ and } n_0 \geq 0 \text{ such that } f(n) \leq c \cdot g(n) \text{ for all } n \geq n_0 \]

Pick \( c = 1000 \) and \( n_0 = 3 \). Then, \( f(n) \leq c \cdot g(n) \) for all \( n \geq n_0 \).
The Change Problem – Running Time Analysis

GreedyChange\( (M, c_1, ..., c_n) \)
1. \( \text{for } i \leftarrow 1 \text{ to } n \) 
2. \( d_i \leftarrow \lfloor M/c_i \rfloor \) 
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DPChange\( (M, c_1, ..., c_n) \)
1. \( \text{for } m \leftarrow 1 \text{ to } M \) 
2. \( \text{minNumCoins}[m] \leftarrow \infty \) 
3. \( \text{for } i \leftarrow 1 \text{ to } n \) 
4. \( \text{minNumCoins}[c_i] \leftarrow 1 \) 
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6. \( \text{for } i \leftarrow 1 \text{ to } n \) 
7. \( \text{if } m > c_i \) 
8. \( \text{minNumCoins}[m] \leftarrow \min(1 + \text{minNumCoins}[m - c_i], \text{minNumCoins}[m]) \) 
9. \( \text{return } \text{minNumCoins}[M] \)

**Number of operations:**
- Line 2: \( 3 = O(1) \)
- Line 3: \( 3 = O(1) \)
- Total: \( 6n = O(n) \)

**Number of operations:**
- Lines 1-2: \( O(M) \)
- Lines 3-4: \( O(n) \)
- Lines 5-8: \( O(Mn) \)
- Total: \( O(M) + O(n) + O(Mn) = O(Mn) \)
Running Time Analysis – Guidelines

• $O(n^a) \subset O(n^b)$ for any positive constants $a < b$

• For any constants $a, b > 0$ and $c > 1$,
  
  $O(a) \subset O(\log n) \subset O(n^b) \subset O(c^n)$

• We can multiply to learn about other functions. For any constants $a, b > 0$ and $c > 1$,
  
  $O(an) = O(n) \subset O(n \log n) \subset O(n n^b) = O(n^{b+1}) \subset O(nc^n)$

• Base of the logarithm is a constant and can be ignored. For any constants $a, b > 1$,
  
  $O(\log_a n) = O(\log_b n / \log_b a) = O(1/(\log_b a) \log_b n) = O(\log_b n)$
Running Time Analysis – Guidelines

- \( O(n^a) \subset O(n^b) \) for any positive constants \( a < b \)

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  \]

<table>
<thead>
<tr>
<th>Big Oh</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>( O(1) )</td>
<td>Constant</td>
</tr>
<tr>
<td>( O(\log n) )</td>
<td>Logarithmic</td>
</tr>
<tr>
<td>( O(n) )</td>
<td>Linear</td>
</tr>
<tr>
<td>( O(n^2) )</td>
<td>Quadratic</td>
</tr>
<tr>
<td>( O(n^c) = O(\text{poly}(n)) )</td>
<td>Polynomial</td>
</tr>
<tr>
<td>( O(2^{\text{poly}(n)}) )</td>
<td>Exponential</td>
</tr>
</tbody>
</table>
Question: What is $O\left(\binom{n}{k}\right)$?
Running Time Analysis – More Examples

**Question:** What is $O\left(\binom{n}{k}\right)$?

- For constant $k > 0$ it holds that $\binom{n}{k} = O(n^k)$

**Question:** What is $O(n!)$?

- Recall that $n! = \prod_{i=1}^{n} i$
Running Time Analysis – More Examples

**Question:** What is $O\left(\binom{n}{k}\right)$?

- For constant $k > 0$ it holds that $\binom{n}{k} = O(n^k)$

- Recall that $n! = \prod_{i=1}^{n} i$

**Question:** What is $O(n!)$?

Stirling’s approximation: $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n = \sqrt{2\pi} \frac{\sqrt{n}}{\exp(n)} n^n = O(n^n) = O(2^n \log n)$

(*) $\sqrt{n} / \exp(n) < 1$ for all $n > 0$

**Question:** Is $n^n = O(n!)$?
Running Time Analysis – More Examples

**Question:** What is $O\left(\binom{n}{k}\right)$?

• For constant $k > 0$ it holds that $\binom{n}{k} = O(n^k)$

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(*): $\sqrt{n} / \exp(n) < 1$ for all $n > 0$

**Question:** Is $n^n = O(n!)$?

**Question:** What is $O(\log(n!))$?
Molecular evolution of FOXP2, a gene involved in speech and language


* Max Planck Institute for Evolutionary Anthropology, Inselstrasse 22, D-04103 Leipzig, Germany
† Wellcome Trust Centre for Human Genetics, University of Oxford, Roosevelt Drive, Oxford OX3 7BN, UK


“Thus, although the FOXP2 protein is extremely conserved among mammals, it acquired two amino-acid changes on the human lineage, at least one of which may have functional consequences. This is an intriguing finding, because FOXP2 is the first gene known to be involved in the development of speech and language.”

<table>
<thead>
<tr>
<th></th>
<th>Human</th>
<th>Chimpanzee</th>
<th>Gorilla</th>
<th>Orangutan</th>
<th>Rhesus</th>
<th>Mouse</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>TS3NTSKASP</td>
<td>PITHHSIVNG</td>
<td>QSSV</td>
<td>SARRD</td>
<td>SSSHEETGAS</td>
<td>HTLYGHGVCX</td>
</tr>
</tbody>
</table>

Figure 1 Alignment of the amino-acid sequences inferred from the FOXP2 cDNA sequences. The polyglutamine stretches and the forkhead domain are shaded. Sites that differ from the human sequence are boxed.

Question: How do we align sequences to identify similarities/differences?
An alignment between two strings \( v \) (of \( m \) characters) and \( w \) (of \( n \) characters) is a two row matrix where the first row contains the characters of \( v \) in order, the second row contains the characters of \( w \) in order, and spaces may be interspersed throughout each.

**Input**

\( v: \) KITTEN \((m = 6)\)

\( w: \) SITTING \((n = 7)\)

**Output**

\( v: \) K - I T T E N -

\( w: \) S I - T T I N G

**Question:** Is this a good alignment?

**Answer:** Count the number of insertions, deletions, and substitutions.
Alignment

An alignment between two strings \( v \) (of \( m \) characters) and \( w \) (of \( n \) characters) is a two row matrix where the first row contains the characters of \( v \) in order, the second row contains the characters of \( w \) in order, and spaces may be interspersed throughout each.

\[
\begin{array}{c}
\text{Input} \\
v: \text{KITTEN} \quad (m = 6) \\
w: \text{SITTING} \quad (n = 7)
\end{array}
\quad \begin{array}{c}
\text{Output} \\
v: \begin{array}{cccccccc}
K & - & I & T & T & E & N & -
\end{array} \\
w: \begin{array}{cccccccc}
S & I & - & T & T & I & N & G
\end{array}
\end{array}
\]

**Question:** Is this a good alignment?

**Answer:** Count the number of insertion, deletions, substitutions.
Edit Distance [Levenshtein, 1966]

**Elementary operations:** insertion, deletions and substitutions of single characters

**Edit Distance problem:** Given strings \( v \in \Sigma^m \) and \( w \in \Sigma^n \), compute the minimum number \( d(v, w) \) of elementary operations to transform \( v \) into \( w \).

\[
d(\text{cat, car}) = 1 \quad d(\text{cat, ate}) = 2 \quad d(\text{cat, are}) = 3
\]
Computing Edit Distance

**Edit Distance problem:** Given strings $v \in \Sigma^m$ and $w \in \Sigma^n$, compute the minimum number $d(v, w)$ of elementary operations to transform $v$ into $w$.

$v$: ATGTTAT...

$w$: AGCGTAC...

Optimal substructure:
Edit distance obtained from edit distance of prefix of string.
Computing Edit Distance – Optimal Substructure

\[ d[i, j] \] is the edit distance of \( v_i \) and \( w_j \),
where \( v_i \) is prefix of \( v \) of length \( i \) and \( w_j \) is prefix of \( w \) of length \( j \)

<table>
<thead>
<tr>
<th>Deletion: ( d[i, j] = d[i - 1, j] + 1 )</th>
<th>( \ldots )</th>
<th>( v_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ldots )</td>
<td>( - )</td>
<td>( \ldots )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Insertion: ( d[i, j] = d[i, j - 1] + 1 )</th>
<th>( \ldots )</th>
<th>( - )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ldots )</td>
<td>( w_j )</td>
<td>( \ldots )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mismatch: ( d[i, j] = d[i - 1, j - 1] + 1 )</th>
<th>( \ldots )</th>
<th>( v_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ldots )</td>
<td>( w_j )</td>
<td>( \ldots )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Match: ( d[i, j] = d[i - 1, j - 1] )</th>
<th>( \ldots )</th>
<th>( v_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ldots )</td>
<td>( w_j )</td>
<td>( \ldots )</td>
</tr>
</tbody>
</table>
Computing Edit Distance – Recurrence

\[ d[i, j] \] is the edit distance of \( \vec{v}_i \) and \( \vec{w}_j \),
where \( \vec{v}_i \) is prefix of \( \vec{v} \) of length \( i \) and \( \vec{w}_j \) is prefix of \( \vec{w} \) of length \( j \)

\[
d[i, j] = \min \begin{cases} 
  d[i - 1, j] + 1, & \text{if } v_i \neq w_j, \\
  d[i, j - 1] + 1, & \text{if } v_i = w_j, \\
  d[i - 1, j - 1] + 1, & \text{if } v_i \neq w_j, \\
  d[i, j - 1] + 1, & \text{if } v_i = w_j.
\end{cases}
\]
Computing Edit Distance – Recurrence

\[ d[i, j] \] is the edit distance of \( v_i \) and \( w_j \),
where \( v_i \) is prefix of \( v \) of length \( i \) and \( w_j \) is prefix of \( w \) of length \( j \)

\[
d[i, j] = \min \begin{cases} 
0, & \text{if } i = 0 \text{ and } j = 0, \\
d[i - 1, j] + 1, & \text{if } i > 0, \\
d[i, j - 1] + 1, & \text{if } j > 0, \\
d[i - 1, j - 1] + 1, & \text{if } i > 0, j > 0 \text{ and } v_i \neq w_j, \\
d[i - 1, j - 1], & \text{if } i > 0, j > 0 \text{ and } v_i = w_j.
\end{cases}
\]
Computing Edit Distance – Dynamic Programming

\[ d[i, j] = \min \begin{cases} 0, & \text{if } i = 0 \text{ and } j = 0, \\ d[i, j - 1] + 1, & \text{if } i > 0, \\ d[i - 1, j] + 1, & \text{if } j > 0, \\ d[i - 1, j - 1] + 1, & \text{if } i > 0, j > 0 \text{ and } v_i \neq w_j, \\ d[i - 1, j - 1], & \text{if } i > 0, j > 0 \text{ and } v_i = w_j. \end{cases} \]
Computing Edit Distance – Dynamic Programming

\[
d[i, j] = \min \begin{cases} 
0, & \text{if } i = 0 \text{ and } j = 0, \\
d[i - 1, j] + 1, & \text{if } i > 0, \\
d[i, j - 1] + 1, & \text{if } j > 0, \\
d[i - 1, j - 1] + 1, & \text{if } i > 0, j > 0 \text{ and } v_i \neq w_j, \\
d[i - 1, j - 1], & \text{if } i > 0, j > 0 \text{ and } v_i = w_j.
\end{cases}
\]

\[\begin{array}{c|cccc}
V & W & A & T & C & G \\
\hline
V & 0 & 1 & 2 & 3 & 4 \\
0 & 0 & & & & \\
1 & & & & & \\
2 & & & & & \\
3 & & & & & \\
4 & & & & & \\
\end{array}\]
### Computing Edit Distance – Dynamic Programming

\[
d[i, j] = \min \begin{cases} 
0, & \text{if } i = 0 \text{ and } j = 0, \\
d[i - 1, j] + 1, & \text{if } i > 0, \\
d[i, j - 1] + 1, & \text{if } j > 0, \\
d[i - 1, j - 1] + 1, & \text{if } i > 0, j > 0 \text{ and } v_i \neq w_j, \\
d[i - 1, j - 1], & \text{if } i > 0, j > 0 \text{ and } v_i = w_j.
\end{cases}
\]

#### Example:

<table>
<thead>
<tr>
<th>V</th>
<th>W</th>
<th>A</th>
<th>T</th>
<th>C</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
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<tr>
<td>2</td>
<td>2</td>
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<td>3</td>
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</tr>
<tr>
<td>4</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- **Deletion** (`v_i`): \( \ldots v_i \) → \( \ldots - \)
- **Insertion** (`w_j`): \( \ldots - \) → \( \ldots w_j \)
- **Mismatch** (`v_i`: `w_j`): \( \ldots v_i \) → \( \ldots w_j \)
- **Match**:

- **0 or 1** match:
  - \( \ldots v_i \) → \( \ldots w_j \)
  - \( \ldots v_i \) → \( \ldots v_i \)

- **1** mismatch:
  - \( \ldots v_i \) → \( \ldots v_i \)
  - \( \ldots v_i \) → \( \ldots w_j \)

- **Mismatch**:
  - \( \ldots v_i \) → \( \ldots v_i \)
  - \( \ldots v_i \) → \( \ldots w_j \)

- **Insertion**:
  - \( \ldots - \) → \( \ldots w_j \)

- **Deletion**:
  - \( \ldots v_i \) → \( \ldots - \)
### Computing Edit Distance – Dynamic Programming

The edit distance between two sequences can be computed using dynamic programming. The edit distance is the minimum number of operations (insertions, deletions, substitutions) required to transform one sequence into another.

#### Dynamic Programming Formula

Let $d[i,j]$ be the edit distance between the first $i$ characters of string $V$ and the first $j$ characters of string $W$. The formula for $d[i,j]$ is given by:

$$d[i,j] = \min\begin{cases} 0, & \text{if } i = 0 \text{ and } j = 0, \\ d[i-1,j] + 1, & \text{if } i > 0, \\ d[i,j-1] + 1, & \text{if } j > 0, \\ d[i-1,j-1] + 1, & \text{if } i > 0, j > 0 \text{ and } v_i \neq w_j, \\ d[i-1,j-1], & \text{if } i > 0, j > 0 \text{ and } v_i = w_j. \\
\end{cases}$$

#### Example

Consider the strings $V = \text{GACT}$ and $W = \text{TGAC}$. The edit distance matrix is:

<table>
<thead>
<tr>
<th></th>
<th>W</th>
<th>A</th>
<th>T</th>
<th>C</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>V</td>
<td></td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
<td>?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>G</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The edit distance between $V$ and $W$ is 1, achieved by substituting $v_2 = C$ with $w_1 = G$.

#### Diagram

A diagram illustrating the dynamic programming approach is shown, with arrows indicating the transitions between states in the matrix.
Computing Edit Distance – Dynamic Programming

\[ d[i, j] = \begin{cases} 
0, & \text{if } i = 0 \text{ and } j = 0, \\
(d[i - 1, j] + 1, & \text{if } i > 0, \\
(d[i, j - 1] + 1, & \text{if } j > 0, \\
(d[i - 1, j - 1] + 1, & \text{if } i > 0, j > 0 \text{ and } v_i \neq w_j, \\
(d[i - 1, j - 1], & \text{if } i > 0, j > 0 \text{ and } v_i = w_j.
\end{cases} \]

<table>
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</thead>
<tbody>
<tr>
<td>V</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
<td></td>
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<tr>
<td>2</td>
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<td>3</td>
<td>3</td>
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</tr>
<tr>
<td>4</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

![Edit Distance Matrix](image)

0 or 1

0 or 1

1

1

1

0 or 1

0 or 1

1

i - 1, j - 1

i - 1, j

i, j - 1

i, j
Computing Edit Distance – Dynamic Programming

\[
d[i, j] = \begin{cases} 
0, & \text{if } i = 0 \text{ and } j = 0, \\
1 + d[i-1, j], & \text{if } i > 0, \\
1 + d[i, j-1], & \text{if } j > 0, \\
1 + d[i-1, j-1], & \text{if } i > 0, j > 0 \text{ and } v_i \neq w_j, \\
1 + d[i-1, j-1], & \text{if } i > 0, j > 0 \text{ and } v_i = w_j. 
\end{cases}
\]

![Dynamic Programming Table](image-url)
Computing Edit Distance – Dynamic Programming

\[
\begin{align*}
\text{d}[i,j] &= \min \left\{ \begin{array}{l}
0, \quad \text{if } i = 0 \text{ and } j = 0, \\
\text{d}[i-1,j] + 1, \quad \text{if } i > 0, \\
\text{d}[i,j-1] + 1, \quad \text{if } j > 0, \\
\text{d}[i-1,j-1] + 1, \quad \text{if } i > 0, j > 0 \text{ and } v_i \neq w_j, \\
\text{d}[i-1,j-1], \quad \text{if } i > 0, j > 0 \text{ and } v_i = w_j.
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
&\text{V} \quad \text{W} \quad \text{A} \quad \text{T} \quad \text{C} \quad \text{G} \\
&\text{V} \quad \begin{array}{c|ccccc}
0 & 0 & 1 & 2 & 3 & 4 \\
1 & 1 & 0 & & & \\
2 & 2 & 1 & & & \\
3 & 3 & & & & \\
4 & 4 & & & & \\
\end{array} \\
\end{align*}
\]

\[
\begin{align*}
&\text{i} - 1, j - 1 \\
&\text{i} - 1, j \\
&\text{i}, j - 1 \\
&\text{i}, j \\
\end{align*}
\]
Computing Edit Distance – Dynamic Programming

The edit distance between two sequences of symbols is the minimum number of basic operations (insertions, deletions, and substitutions) required to transform one sequence into the other. Dynamic programming is a useful technique for computing the edit distance between two sequences.

Let's consider two sequences $V = v_1 v_2 \ldots v_i$ and $W = w_1 w_2 \ldots w_j$. The edit distance $d[i, j]$ can be computed as follows:

$$d[i, j] = \begin{cases} 0, & \text{if } i = 0 \text{ and } j = 0, \\ d[i - 1, j] + 1, & \text{if } i > 0, \\ d[i, j - 1] + 1, & \text{if } j > 0, \\ d[i - 1, j - 1], & \text{if } i > 0, j > 0 \text{ and } v_i = w_j, \\ d[i - 1, j - 1] + 1, & \text{if } i > 0, j > 0 \text{ and } v_i \neq w_j. \end{cases}$$

We can visualize this using a dynamic programming table and a graph:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>T</th>
<th>C</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>V</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
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<tr>
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<td>2</td>
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<td>3</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For example, consider the sequences $V = \ldots v_i \ldots$ and $W = \ldots w_j \ldots$.

```
... v_i ...
... - ...
... w_j ...
```

- **Deletion**: Subtract 1 from the previous row.
- **Insertion**: Subtract 1 from the previous column.
- **Mismatch**: Use the diagonal value.
- **Match**: Use the diagonal value.

The graph shows how the values are computed from previous cells.
Computing Edit Distance – Dynamic Programming

\[
d[i, j] = \min \begin{cases} 
0, & \text{if } i = 0 \text{ and } j = 0, \\
2, & \text{if } i > 0, \\
3, & \text{if } j > 0, \\
4, & \text{if } i > 0 \text{ and } j > 0 \text{ and } v_i \neq w_j, \\
5, & \text{if } i > 0 \text{ and } j > 0 \text{ and } v_i = w_j.
\end{cases}
\]

**Table:**

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>T</th>
<th>C</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>V</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
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<td>2</td>
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<td>3</td>
<td>3</td>
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</tr>
<tr>
<td>4</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Diagram:**

\[
\begin{align*}
i - 1, j - 1 & \quad 0 \text{ or } 1 \\
i - 1, j & \quad 1 \\
i, j - 1 & \quad 1 \\
i, j & \quad \text{match}
\end{align*}
\]
Computing Edit Distance – Dynamic Programming

\[
d[i, j] = \min\left\{ \begin{array}{ll}
0, & \text{if } i = 0 \text{ and } j = 0, \\
 d[i-1, j] + 1, & \text{if } i > 0, \\
 d[i, j-1] + 1, & \text{if } j > 0, \\
 d[i-1, j-1] + 1, & \text{if } i > 0, j > 0 \text{ and } v_i \neq w_j, \\
 d[i-1, j-1], & \text{if } i > 0, j > 0 \text{ and } v_i = w_j.
\end{array} \right.
\]

- **deletion**
- **insertion**
- **mismatch**
- **match**

\[
\begin{array}{c|cccc}
\text{W} & \text{A} & \text{T} & \text{C} & \text{G} \\
\hline
\text{V} & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 1 & 2 & 3 & 4 \\
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 \\
\end{array}
\]
Computing Edit Distance – Dynamic Programming

\[ d[i, j] = \min \begin{cases} 
0, & \text{if } i = 0 \text{ and } j = 0, \\
|d[i-1, j]| + 1, & \text{if } i > 0, \\
|d[i, j-1]| + 1, & \text{if } j > 0, \\
|d[i-1, j-1]| + 1, & \text{if } i > 0, j > 0 \text{ and } v_i \neq w_j, \\
|d[i-1, j-1]|, & \text{if } i > 0, j > 0 \text{ and } v_i = w_j.
\end{cases} \]
Computing Edit Distance – Dynamic Programming

\begin{align*}
\text{W} & \quad \text{A} & \quad \text{T} & \quad \text{C} & \quad \text{G} \\
\text{V} & \quad 0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 \\
0 & \quad 0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 \\
1 & \quad 1 & \quad 0 & \quad 1 & \quad 2 & \quad 3 \\
2 & \quad 2 & \quad 1 & \quad 0 & \quad 1 & \quad 2 \\
3 & \quad 3 & \quad 2 & \quad 1 & \quad 1 & \quad 1 \\
4 & \quad 4 & \quad 3 & \quad 2 & \quad 2 & \quad 2 \\
\end{align*}

\[ d[i, j] = \min \begin{cases} 
0, & \text{if } i = 0 \text{ and } j = 0, \\
\delta[i - 1, j] + 1, & \text{if } i > 0, \\
\delta[i, j - 1] + 1, & \text{if } j > 0, \\
\delta[i - 1, j - 1] + 1, & \text{if } i > 0, j > 0 \text{ and } v_i \neq w_j, \\
\delta[i - 1, j - 1], & \text{if } i > 0, j > 0 \text{ and } v_i = w_j. 
\end{cases} \]

- **Deletion**: \( \ldots \quad v_i \)
- **Insertion**: \( \ldots \quad - \)
- **Mismatch**: \( \ldots \quad v_i \quad \ldots \quad w_j \)
- **Match**: \( \ldots \quad w_j \)

\[ \begin{align*}
\text{V} & \quad \text{W} \\
0 & \quad 0 \\
1 & \quad \delta[i - 1, j] \\
2 & \quad \delta[i, j - 1] \\
3 & \quad \delta[i - 1, j - 1] \\
4 & \quad \delta[i - 1, j - 1] \\
\end{align*} \]

\[ \begin{align*}
\text{V} & \quad \text{A} \\
0 & \quad 0 \\
1 & \quad \delta[i - 1, j] \\
2 & \quad \delta[i, j - 1] \\
3 & \quad \delta[i - 1, j - 1] \\
4 & \quad \delta[i - 1, j - 1] \\
\end{align*} \]

\[ \begin{align*}
\text{V} & \quad \text{T} \\
0 & \quad 0 \\
1 & \quad \delta[i - 1, j] \\
2 & \quad \delta[i, j - 1] \\
3 & \quad \delta[i - 1, j - 1] \\
4 & \quad \delta[i - 1, j - 1] \\
\end{align*} \]

\[ \begin{align*}
\text{V} & \quad \text{C} \\
0 & \quad 0 \\
1 & \quad \delta[i - 1, j] \\
2 & \quad \delta[i, j - 1] \\
3 & \quad \delta[i - 1, j - 1] \\
4 & \quad \delta[i - 1, j - 1] \\
\end{align*} \]

\[ \begin{align*}
\text{V} & \quad \text{G} \\
0 & \quad 0 \\
1 & \quad \delta[i - 1, j] \\
2 & \quad \delta[i, j - 1] \\
3 & \quad \delta[i - 1, j - 1] \\
4 & \quad \delta[i - 1, j - 1] \\
\end{align*} \]

\[ \begin{align*}
\text{V} & \quad \text{T} \\
0 & \quad 0 \\
1 & \quad \delta[i - 1, j] \\
2 & \quad \delta[i, j - 1] \\
3 & \quad \delta[i - 1, j - 1] \\
4 & \quad \delta[i - 1, j - 1] \\
\end{align*} \]
Computing Edit Distance – Dynamic Programming

\[
\begin{align*}
\text{deletion} & \quad \text{insertion} & \quad \text{mismatch} & \quad \text{match} \\
\ldots\text{v}_i & \quad \ldots\text{w}_j & \quad \ldots\text{v}_i & \quad \ldots\text{w}_j \\
\ldots\text{-} & \quad \ldots\text{-} & \quad \ldots\text{-} & \quad \ldots\text{-} \\
\end{align*}
\]

\[
d[i, j] = \min \begin{cases} 
0, & \text{if } i = 0 \text{ and } j = 0, \\
\text{d}[i-1, j] + 1, & \text{if } i > 0, \\
\text{d}[i, j-1] + 1, & \text{if } j > 0, \\
\text{d}[i-1, j-1] + 1, & \text{if } i > 0, \ j > 0 \text{ and } \text{v}_i \neq \text{w}_j, \\
\text{d}[i-1, j-1], & \text{if } i > 0, \ j > 0 \text{ and } \text{v}_i = \text{w}_j.
\end{cases}
\]

\[
\begin{array}{cccccc}
V & 0 & 1 & 2 & 3 & 4 \\
\hline
W & \begin{array}{cccccc}
0 & 0 & 1 & 2 & 3 & 4 \\
A & 1 & 1 & 0 & 1 & 2 & 3 \\
T & 2 & 2 & 1 & 0 & 1 & 2 \\
G & 3 & 3 & 2 & 1 & 1 & 1 \\
T & 4 & 4 & 3 & 2 & 2 & 2 \\
\end{array}
\end{array}
\]

\[
\begin{array}{cccccc}
\text{A} & \text{T} & \text{-} & \text{G} & \text{T} \\
\text{A} & \text{T} & \text{C} & \text{G} & \text{-} \\
\text{A} & \text{T} & \text{G} & \text{T} & \text{-} \\
\text{A} & \text{T} & \text{C} & \text{G} & \text{-} \\
\end{array}
\]
Computing Edit Distance – Running Time

For each \((m + 1)\times(n + 1)\) entry:
- 3 addition operations
- 1 comparison operation
- 1 minimum operation

Running time: \(O(mn)\) time

\[
d[i, j] = \min \begin{cases} 
0, & \text{if } i = 0 \text{ and } j = 0, \\
d[i - 1, j] + 1, & \text{if } i > 0, \\
d[i, j - 1] + 1, & \text{if } j > 0, \\
d[i - 1, j - 1] + 1, & \text{if } i > 0, j > 0 \text{ and } v_i \neq w_j, \\
d[i - 1, j - 1], & \text{if } i > 0, j > 0 \text{ and } v_i = w_j.
\end{cases}
\]
### Computing Edit Distance – Running Time

For each \((m + 1) \times (n + 1)\) entry:
- 3 addition operations
- 1 comparison operation
- 1 minimum operation

Running time: \(O(mn)\) time

---

<table>
<thead>
<tr>
<th></th>
<th>W</th>
<th>A</th>
<th>T</th>
<th>C</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>V</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
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<tr>
<td>1</td>
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<td>4</td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

The edit distance matrix is computed with the following formula:

\[
d[i, j] = \min\left\{ \begin{align*}
0, & \quad \text{if } i = 0 \text{ and } j = 0, \\
\min(0, d[i - 1, j] + 1, d[i, j - 1] + 1, d[i - 1, j - 1] + 1), & \quad \text{if } i > 0, j > 0 \text{ and } v_i \neq w_j, \\
\min(0, d[i - 1, j], d[i, j - 1]), & \quad \text{if } i > 0, j > 0 \text{ and } v_i = w_j.
\end{align*} \right\
\]

---

The diagram illustrates the edit distance computation, where:
- \(d[i - 1, j - 1]\) is the match case.
- \(d[i - 1, j] + 1\) is the deletion case.
- \(d[i, j - 1] + 1\) is the insertion case.
- \(d[i - 1, j - 1] + 1\) is the mismatch case.

The columns represent the characters A, T, C, G, and the rows represent the characters V, W. The values represent the minimum edit distance for each pair of characters.
Computing Edit Distance – Your turn!

\[ d[i,j] = \begin{cases} 
0, & \text{if } i = 0 \text{ and } j = 0, \\
0, & \text{if } j = 0, \\
d[i-1,j] + 1, & \text{if } i > 0, \\
d[i,j-1] + 1, & \text{if } j > 0, \\
d[i-1,j-1] + 1, & \text{if } i > 0, j > 0 \text{ and } v_i \neq w_j, \\
d[i-1,j-1], & \text{if } i > 0, j > 0 \text{ and } v_i = w_j.
\end{cases} \]

\[ d(\text{cat, car}) = \]
\[ d(\text{cat, ate}) = \]
\[ d(\text{cat, are}) = \]
Change Problem and Edit distance

Make \( M \) cents using minimum number of 1, 3 and 5 cent coins.

<table>
<thead>
<tr>
<th>Value</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min # coins</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

- Both have optimal substructure and can be solved using dynamic programming
- These are examples of a more general problem!
Review of Graph Theory

- Graph $G = (V, E)$
- Vertices $V = \{v_1, \ldots, v_n\}$
- Edges $E = \{(v_i, v_j), \ldots\}$
Review of Graph Theory

- Directed Graph $G = (V, E)$
- Vertices $V = \{v_1, ..., v_n\}$
- Directed edges $E = \{(v_i, v_j), ...\}$
Review of Graph Theory

- Directed Graph $G = (V, E)$
- Vertices $V = \{v_1, \ldots, v_n\}$
- Directed edges $E = \{(v_i, v_j), \ldots\}$
- Path is a sequence of vertices and edges that connect them
Review of Graph Theory

- Directed Graph $G = (V, E)$
- Vertices $V = \{v_1, \ldots, v_n\}$
- Directed edges $E = \{(v_i, v_j), \ldots\}$
- Path is a sequence of vertices and edges that connect them
- Edges can be weighted
A tourist in Manhattan wants to visit the maximum number of attractions (*) by traveling on a path (only eastward and southward) from start to end.
A tourist in Manhattan wants to visit the maximum number of attractions (*) by traveling on a path (only eastward and southward) from start to end.

May be more than 1 attraction on a street. Add weights!
Manhattan Tourist Problem (MTP):
Given a weighted, directed grid graph $G$ with two vertices “begin” and “end”, find the maximum weight path in $G$ from “begin” to “end”. 
Manhattan Tourist Problem – Exhaustive Algorithm

Check all paths

Question: How many paths?
Manhattan Tourist Problem – Greedy Algorithm

promising start, but leads to bad choices!

better path!
Manhattan Tourist Problem – Optimal Substructure

begin

\[
\begin{array}{cccc}
5 & 1 & 2 & 5 \\
3 & 2 & 1 & 5 \\
2 & 3 & 10 & 5 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 \\
end
\end{array}
\]

best score to this point

\[
\begin{align*}
18 &
\end{align*}
\]

best score to end

\[
22
\]
Manhattan Tourist Problem – Optimal Substructure

$s[i, j]$ is the best score for path to coordinate $(i, j)$

**Question:** What is the recurrence?

- $w[(i-1, j), (i, j)]$ weight of street between $(i-1, j)$ and $(i, j)$
- $w[(i, j-1), (i, j)]$ weight of street between $(i, j-1)$ and $(i, j)$
Manhattan Tourist Problem – Optimal Substructure

$s[i, j]$ is the best score for path to coordinate $(i, j)$

$$s[i, j] = \max \begin{cases} 
0, & \text{if } i = 0 \text{ and } j = 0, \\
 s[i - 1, j] + w[(i - 1, j), (i, j)], & \text{if } i > 0, \\
 s[i, j - 1] + w[(i, j - 1), (i, j)], & \text{if } j > 0.
\end{cases}$$

- $w[(i - 1, j), (i, j)]$ weight of street between $(i - 1, j)$ and $(i, j)$
- $w[(i, j - 1), (i, j)]$ weight of street between $(i, j - 1)$ and $(i, j)$
MTP – Solving Recurrence using Dynamic Programming

\[ s[i, j] \text{ is the best score for path to coordinate } (i, j) \]

\[ s[i, j] = \max \begin{cases} 
0, & \text{if } i = 0 \text{ and } j = 0, \\
 s[i - 1, j] + w[(i - 1, j), (i, j)], & \text{if } i > 0, \\
 s[i, j - 1] + w[(i, j - 1), (i, j)], & \text{if } j > 0.
\end{cases} \]

- \( w[(i - 1, j), (i, j)] \) weight of street between \((i - 1, j)\) and \((i, j)\)
- \( w[(i, j - 1), (i, j)] \) weight of street between \((i, j - 1)\) and \((i, j)\)
MTP – Solving Recurrence using Dynamic Programming

\[ s[i, j] \] is the best score for path to coordinate \((i, j)\)

\[
s[i, j] = \begin{cases} 
0, & \text{if } i = 0 \text{ and } j = 0, \\
\max\left\{ s[i - 1, j] + w[(i - 1, j), (i, j)], s[i, j - 1] + w[(i, j - 1), (i, j)] \right\}, & \text{otherwise,}
\end{cases}
\]

- \( w[(i - 1, j), (i, j)] \) weight of street between \((i - 1, j)\) and \((i, j)\)
- \( w[(i, j - 1), (i, j)] \) weight of street between \((i, j - 1)\) and \((i, j)\)
MTP – Solving Recurrence using Dynamic Programming

$s[i,j]$ is the best score for path to coordinate $(i,j)$

\[
s[i,j] = \begin{cases} 
0, & \text{if } i = 0 \text{ and } j = 0, \\
\max(s[i-1,j] + w[(i-1, j), (i, j)], s[i,j-1] + w[(i, j-1), (i, j)]) & \text{if } i > 0, j > 0.
\end{cases}
\]

- $w[(i-1, j), (i, j)]$ weight of street between $(i-1, j)$ and $(i, j)$
- $w[(i, j-1), (i, j)]$ weight of street between $(i, j-1)$ and $(i, j)$
MTP – Solving Recurrence using Dynamic Programming

$s[i, j]$ is the best score for path to coordinate $(i, j)$

$$s[i, j] = \max \begin{cases} 
0, & \text{if } i = 0 \text{ and } j = 0, \\
(s[i-1, j] + \text{w}[(i-1, j), (i, j)]) & \text{if } i > 0, \\
(s[i, j-1] + \text{w}[(i, j-1), (i, j)]) & \text{if } j > 0.
\end{cases}$$

- $\text{w}[(i - 1, j), (i, j)]$ weight of street between $(i - 1, j)$ and $(i, j)$
- $\text{w}[(i, j - 1), (i, j)]$ weight of street between $(i, j - 1)$ and $(i, j)$
MTP – Solving Recurrence using Dynamic Programming

\[ s[i,j] \] is the best score for path to coordinate \((i,j)\)

\[
    s[i,j] = \begin{cases} 
        0, & \text{if } i = 0 \text{ and } j = 0, \\
        s[i-1, j] + w[(i-1,j),(i,j)], & \text{if } i > 0, \\
        s[i, j-1] + w[(i,j-1),(i,j)], & \text{if } j > 0.
    \end{cases}
\]

- \(w[(i-1,j),(i,j)]\) weight of street between \((i-1,j)\) and \((i,j)\)
- \(w[(i,j-1),(i,j)]\) weight of street between \((i,j-1)\) and \((i,j)\)
MTP – Solving Recurrence using Dynamic Programming

\[ s[i, j] \] is the best score for path to coordinate \((i, j)\)

\[ s[i, j] = \max \begin{cases} 
0, & \text{if } i = 0 \text{ and } j = 0, \\
 s[i-1, j] + w[(i-1, j), (i, j)], & \text{if } i > 0, \\
 s[i, j-1] + w[(i, j-1), (i, j)], & \text{if } j > 0.
\end{cases} \]

- \( w[(i-1, j), (i, j)] \) weight of street between \((i-1, j)\) and \((i, j)\)
- \( w[(i, j-1), (i, j)] \) weight of street between \((i, j-1)\) and \((i, j)\)
MTP – Solving Recurrence using Dynamic Programming

\[ s[i, j] \text{ is the best score for path to coordinate } (i, j) \]

\[
s[i, j] = \max \begin{cases} 
0, & \text{if } i = 0 \text{ and } j = 0, \\
\min(s[i-1, j] + w[(i-1, j), (i, j)], s[i, j-1] + w[(i, j-1), (i, j)]) & \text{if } i > 0, \text{ and } j > 0.
\end{cases}
\]

- \( w[(i - 1, j), (i, j)] \) weight of street between \((i - 1, j)\) and \((i, j)\)
- \( w[(i, j - 1), (i, j)] \) weight of street between \((i, j - 1)\) and \((i, j)\)
MTP – Solving Recurrence using Dynamic Programming

Let \( m \) be the number of rows and \( n \) be the number of columns.

**Running time:** \( O(mn) \)

**Question:** Implementation?
Manhattan Is Not a Perfect Grid

What about diagonals?

\[
s[B] = \max \left\{ s[A_1] + w[A_1, B], s[A_2] + w[A_2, B], s[A_3] + w[A_3, B] \right\}.
\]
Manhattan Is Not a Perfect Grid, It’s a Directed Graph

\[ G = (V, E) \] is a directed acyclic graph (DAG) with nonnegative edges weights \( w : E \rightarrow \mathbb{R}^+ \)

- \( s[0, 0] = 0 \)
- \( s[i, j] = \max_{(i', j') \in \text{pred}(i, j)} \{ s[i', j'] + w[(i', j'), (i, j)] \} \)

Each edge is evaluated once: \( O(|E|) \) time
Dynamic Programming as a Graph Problem

**Manhattan Tourist Problem:**
Every path in directed graph is a possible tourist path. Find **maximum weight path**. Running time: $O(mn) = O(|E|)$

**Change Problem:** Make $M$ cents using minimum number of coins $c = (1, 3, 5)$. Every path in directed graph is a possible change. Find **shortest path**. Running time: $O(Mn) = O(|E|)$
What About the Edit Distance Problem?

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**Edit Distance problem:** Given strings $v \in \Sigma^m$ and $w \in \Sigma^n$, compute the minimum number $d(v, w)$ of elementary operations to transform $v$ into $w$. 

- deletion
- insertion
- mismatch
- match
What About the Edit Distance Problem?

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**Edit Distance problem:** Given strings \( \mathbf{v} \in \Sigma^m \) and \( \mathbf{w} \in \Sigma^n \), compute the minimum number \( d(\mathbf{v}, \mathbf{w}) \) of elementary operations to transform \( \mathbf{v} \) into \( \mathbf{w} \).

**Edit graph** is a weighed, directed grid graph \( G = (V, E) \) with source vertex \((0, 0)\) and target vertex \((m, n)\). Each edge \((i, j)\) has weight \([i, j]\) corresponding to edit cost: deletion (1), insertion (1), mismatch (1) and match (0).
What About the Edit Distance Problem?

**Edit graph** is a weighed, directed grid graph $G = (V, E)$ with source vertex $(0, 0)$ and target vertex $(m, n)$. Each edge $(i, j)$ has weight $[i, j]$ corresponding to edit cost: deletion (1), insertion (1), mismatch (1) and match (0).

Alignment is a path from $(0, 0)$ to $(m, n)$
What About the Edit Distance Problem?

**Edit Distance problem**: Given edit graph $G = (V, E)$, with edge weights $c : E \rightarrow \{0,1\}$. Find shortest path from $(0, 0)$ to $(m, n)$.

**Edit graph** is a weighed, directed grid graph $G = (V, E)$ with source vertex $(0, 0)$ and target vertex $(m, n)$. Each edge $(i, j)$ has weight $[i, j]$ corresponding to edit cost: deletion (1), insertion (1), mismatch (1) and match (0).

Alignment is a path from $(0, 0)$ to $(m, n)$.
Shortest Path vs Longest Path

• Change graph, edit graph and the MTP grid are directed graphs $G$.

• Change problem and Edit Distance problem are minimization problems.
• Find shortest path in $G$ from source to sink.

• Manhattan Tourist problem is a maximization problem.
• Find longest path in $G$ from source to sink.
Shortest Path vs Longest Path

• Shortest path in directed graphs can be found efficiently (Dijkstra, Bellman-Ford, Floyd-Warshall algorithms)

• Longest path in direct graphs cannot be found efficiently (NP-hard).

• Change graph, edit graph and MTP grid graph are directed acyclic graphs (DAGs).

• No directed cycles.

• Longest path problem in a DAG can be solved efficiently by dynamic programming

Question: What’s the relation between absence of directed cycles and optimal substructure?
Weighted Edit Distance

\[ d[i, j] \] is the edit distance of \( v_i \) and \( w_j \), where \( v_i \) is prefix of \( v \) of length \( i \) and \( w_j \) is prefix of \( w \) of length \( j \)

\[
\begin{align*}
    d[i, j] &= \min \left\{ 
        d[i - 1, j] + 1, \\
        d[i, j - 1] + 1, \\
        d[i - 1, j - 1] + 1, \quad \text{if } v_i \neq w_j, \\
        d[i - 1, j - 1], \quad \text{if } v_i = w_j.
    \right\}
\end{align*}
\]

Replace +1 with different penalties for different types of edits.
Summary

1. Change problem
2. Review of running time analysis
3. Edit distance
4. Review elementary graph theory
5. Manhattan Tourist problem
6. Longest/shortest paths in DAGs

Reading:
• Jones and Pevzner. Chapters 2.7-2.9 and 6.1-6.4
• Lecture notes
Sources

• CS 362 by Layla Oesper (Carleton College)
• CS 1810 by Ben Raphael (Brown/Princeton University)
• An Introduction to Bioinformatics Algorithms book (Jones and Pevzner)
• http://bioalgorithms.info/